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M.Sc DEGREE (CSS) EXAMINATION, MARCH 2026

Fourth Semester

M Sc MATHEMATICS

Core Courses - ME010402 - ANALYTIC NUMBER THEORY

2019 Admission Onwards

0DBCA606

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. State and prove the product formula for $\phi(n)$.
2. Given $f(1) = 1$.Then prove that f is multiplicative if and only if $f(p_1^{a_1} \dots p_r^{a_r}) = f(p_1)^{a_1} \dots f(p_r)^{a_r}$ for all primes p_i and all integers $a_i \geq 1$.
3. If α is a completely multiplicative function then prove that $G(x) = \sum_{n \leq x} \alpha(n) F(\frac{x}{n})$ if and only if $F(x) = \sum_{n \leq x} \mu(n)\alpha(n) G(\frac{x}{n})$.
4. State Shapiro's Taubarian Theorem.
5. Prove that $\forall\, x \geq 1, \sum_{n \leq x} \vartheta(\frac{x}{n}) = x \log x - x + O(x)$.
6. For a given modulus m prove the following.
   (a) $\hat{a} = \hat{b}$ if and only if $a \equiv b (mod\, m)$
   (b) Two integers x and y are in the same residue class if and only if $x \equiv y (mod\, m)$
7. Prove that if a prime p does not divide a then $a^{p-1} \equiv 1 (mod\, p)$.
8. For any prime p , prove that $(p-1)! \equiv -1 (mod\, p)$.
9. Define quadratic residues. Find the quadratic residues for p=7 .
10. (a)Define $exp_m(a)$.
    (b)Let $m \geq 1$ and $(a, m) = 1$. Prove that the numbers $1, a, a^2, \dots, a^{f-1}$ are incongruent mod m, where, $f = exp_m(a)$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove the following for $x \geq 1, (a) \sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$ *if* $s > 1$.

    (b) $\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$ *if* $\alpha \geq 0$.

12. Prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)$ *for* $x \geq 1$.

    Also prove that the average order of $\sigma_1(n)$ *is* $\frac{\pi^2 n}{12}$.

13. Let $a(n)$ be any arithmetical function and $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Also f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Prove that $\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$.

14. Prove that $\frac{1}{6}n \log n < P_n < 12(\log n + n \log \frac{12}{e}), \forall\, n \geq 1$ where P_n is the the n^{th} prime.

15. (a) if $a \equiv b(mod m)$ then prove that $(a, m) = (b, m)$.

    (b) State and prove cancellation law.

16. Assume $m_1, \ldots, m_r$ are positive integers, relatively prime in pairs. Let $b_1, \ldots, b_r$ be arbitrary integers. Then prove that the system of congruences $x \equiv b_1(mod m_1), \ldots, x \equiv b_r(mod m_r)$ has exactly one solution modulo $m_1, \ldots, m_r$.

17. Derive a formula for finding $(2|p)$ when p is an odd prime

18. State and prove a necessary and sufficient condition for $exp_m(a^k) = exp_m(a)$

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (a)Prove that for every $x \geq 1, [x]! = \Pi_{p \leq x} p^{\alpha(p)}$, where the product is extended over all primes $\leq x$, *and* $\alpha(p) = \sum_{m=1}^{\infty} [\frac{x}{p^m}]$.

    (b) If $x \geq 2$ prove that $log[x]! = x \log x - x + O(\log x)$.

    (c) Prove that $\sum_{p \leq x} [\frac{x}{p}] log\, p = x \log x + O(x)$ for $x \geq 2$, where the sum is extended over all primes $\leq x$.

20. State and prove any three equivalent forms of the prime number theorem.

21. (a) Let f be a polynomial with integer coefficents, let $m_1, m_2, ..., m_r$ be positive integer relatively prime in pairs, and let $m = m_1 m_2 ... m_r$. Prove that the congruence $f(x) \equiv 0 (mod m)$ has a solution if and only each of the congruence $f(x) \equiv 0 (mod m_i)$, i=1,,2,…,r , has a solution.

Also show that if v(m) and $v(m_i)$ denote the number of solutions of $f(x) \equiv 0 (mod m)$ and $f(x) \equiv 0 (mod m_i)$, i=1,,2,…,r respectively, then $v(m) = v(m_1)v(m_2)...v(m_r)$.

(b) Prove that for a given any integer k > 0 there exists a lattice point (a , b) such that none of the lattice points (a+r, b+s), 0<r, s < k is visible from the orgin.

22. (a)State Gauss' lemma.

(b) Let m be the number defined in Gauss' lemma. Then prove that

$m \equiv \sum_{t=1}^{\frac{p-1}{2}} [\frac{tn}{p}] + (n-1)\frac{p^2-1}{8} (mod 2)$.

(2×5=10 weightage)